Horo-Convex Optimization on Hadamard Manifolds

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EPFL

Setting

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$\mathcal{M}$ is a Hadamard manifold

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- Horo-convexity, its nice properties and limitations
  - Nec&suff interpolation conditions, complexity guarantees, etc.
  - Generalization of (Euclidean) convexity

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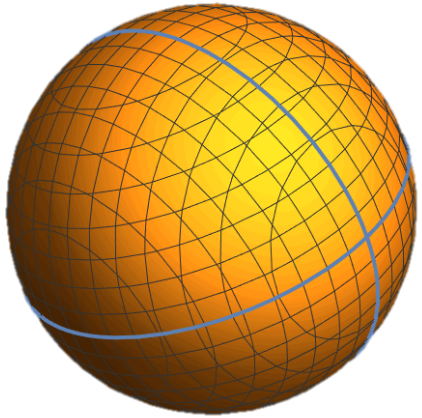

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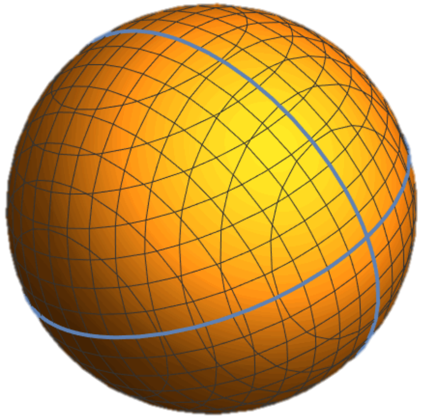

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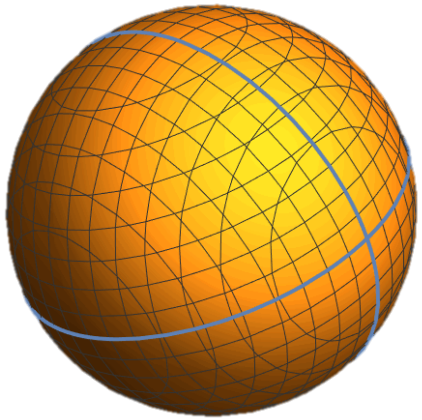

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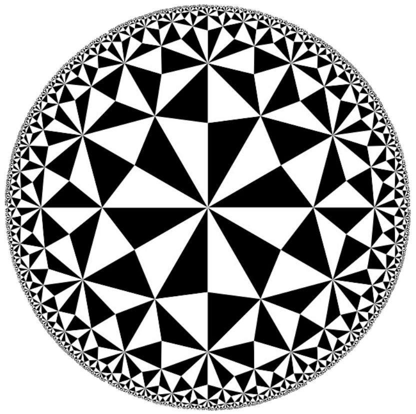

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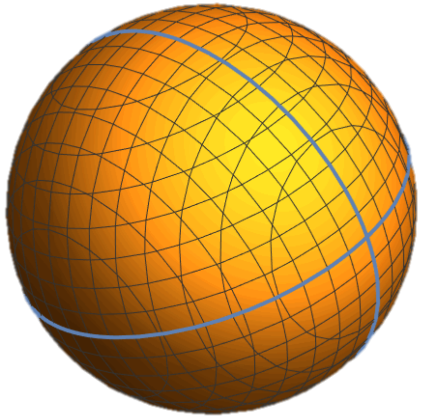

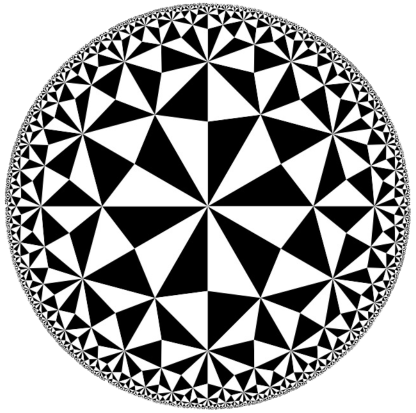

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Positive definite matrices: $\mathcal{M} = \{\Sigma \in \mathbf{R}^{d\times d} : \Sigma = \Sigma^\top \text{ and } \Sigma \succ 0\}$ with affine-invariant metric $\langle X, Y\rangle_\Sigma = \mathrm{Tr}(\Sigma^{-1} X \Sigma^{-1} Y)$.

Applications of g-convexity

Minimal enclosing ball (MEB) of set $C \subseteq \mathcal{M}$ (e.g., Arnaudon + Nielsen)

$$\min_{x \in \mathcal{M}} \max_{y \in C} \operatorname{dist}(x, y)$$

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Operator scaling, tensor scaling, Horn's problem, Brascamp-Lieb constants, …

- "Towards a theory of non-commutative optimization" -- Bürgisser, Franks, Garg, Oliveira, Walter, Wigderson

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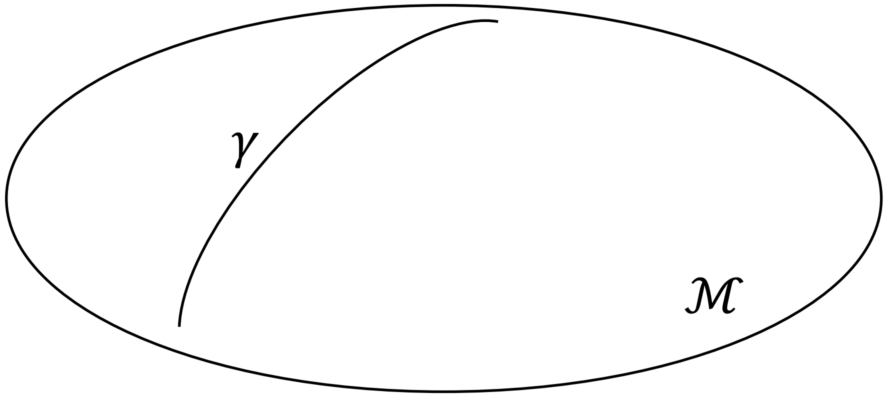

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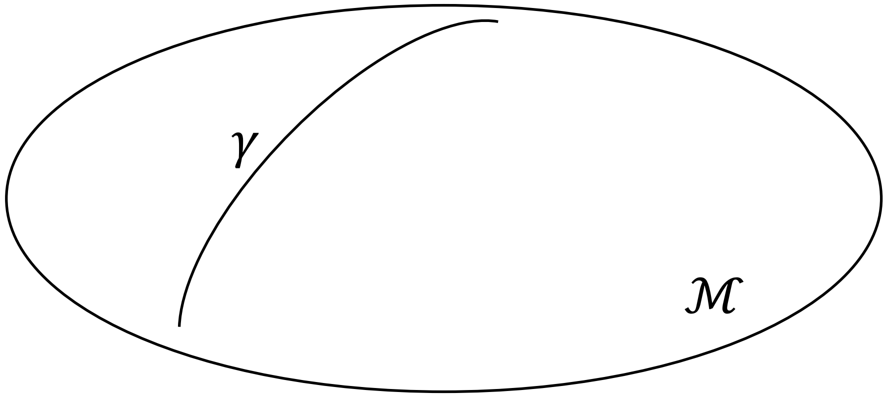

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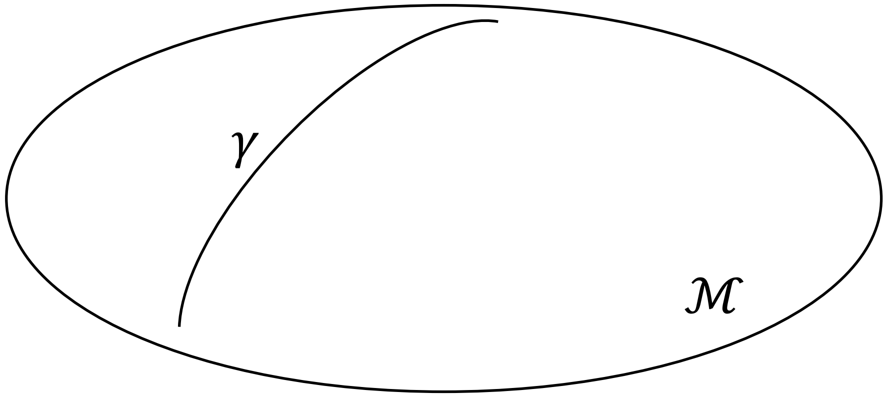

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Alternative characterization: for all $x, y \in \mathcal{M}$

$$f(y) \geq f(x) + \langle \nabla f(x), \exp_x^{-1}(y) \rangle$$

Exponential map: for $x \in \mathcal{M}$ and $v \in \mathrm{T}_x\mathcal{M}$,

$$\gamma(t) = \exp_x(tv) \text{ is the geodesic with } \gamma(0) = x,\ \gamma'(0) = v.$$

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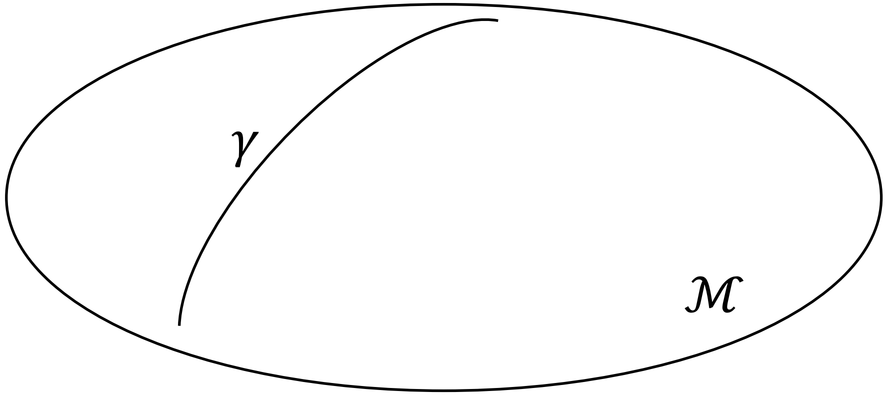

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Complexity of all algorithms depend on curvature ζ of the manifold, e.g.:

- Sub-gradient descent: $O(\zeta/\epsilon^2)$
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(To find a point with relative accuracy $\epsilon \in (0,1)$.)

$$\zeta = \frac{r\sqrt{-K}}{\tanh(r\sqrt{-K})}$$

K = sectional curvature

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Next:

- We propose and study a function class which generalizes Euclidean convexity, and resolves all these issues
- Downside: the function class is smaller, so covers less examples

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Horo-convexity

Starting observations:

- Each "basis" function $y \mapsto f(x) + \langle \nabla f(x), \exp_x^{-1}(y) \rangle$ is NOT g-convex
- μ-strongly convex functions are suprema of squared distance functions:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$

$$= f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2 + \frac{\mu}{2} \|y - x^{++}\|^2 \qquad \text{where } x^{++} = x - \frac{1}{\mu} \nabla f(x).$$

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$\mu = 0$? Take the limit!

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"Horoballs and the subgradient method" -- Lewis, Lopez-Acedo, Nicolae 2024

"Horoball Hulls and Extents in Positive Definite Space" -- Fletcher et al. 2011

"Abstract convexity and global optimization" -- Rubinov

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Big caveat: not closed under addition! (unlike g-convex functions)

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with exactly the same rates, and same step sizes!

Subgradient descent $x_{k+1} = \exp_{x_k}(-\eta_k g_k)$: $O(1/\epsilon^2)$
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For **MEB**: get the best known rate of $\min\left\{\frac{n}{\epsilon^2}, \frac{n\sqrt{\log(n)}\cdot poly(\zeta)}{\epsilon}\right\}$ to reach relative accuracy ϵ.

Interpolation: g-convex

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A convex function interpolates $(F_i, x_i, g_i)_{i=1}^N$ if and only if

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For g-convex functions the analogous naïve necessary conditions are *not* sufficient for interpolation even for just 3 points:

- There exists $(F_i, x_i, g_i)_{i=1}^{3}$ such that $F_j \geq F_i + \left\langle g_i, \log_{x_j}(x_j) \right\rangle$ for all i, j, yet this data cannot be interpolated by a g-convex function.

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Nec&suff interpolation conditions for L-smooth horo-convex function:

$$F_j \geq F_i + \frac{\|g_i\|^2}{2L} + \frac{\|g_j\|^2}{2L} + B_{x_i, g_i}\left(\exp_{x_j}\left(-\frac{g_j}{L}\right)\right) \quad \text{for all } i, j.$$

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Nec&suff interpolation conditions for horo-convex functions:

$$F_j \geq F_i + B_{x_i,g_i}(x_j) \text{ for all } i, j$$

Nec&suff interpolation conditions for L-smooth horo-convex function:

$$F_j \geq F_i + \frac{\|g_i\|^2}{2L} + \frac{\|g_j\|^2}{2L} + B_{x_i,g_i}\left(\exp_{x_j}\left(-\frac{g_j}{L}\right)\right) \quad \text{for all } i, j.$$

Surprising because no known notion of Fenchel conjugate!

Obtained through the Moreau envelope.

Conclusions and future directions

Summary:

- Proposed and studied a generalization of convex functions to Hadamard manifolds
- Geometrically very natural
- Curvature independent rates, and faster algos for MEB
- Nec&suff interpolation conditions

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- Proposed and studied a generalization of convex functions to Hadamard manifolds
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Questions:

- Duality? (boundary at infinity …)
- PEP for horo-convex functions? (to help answer lower bound questions?)
- Inner characterization of horo-convexity?
- Original motivation: Operator scaling, tensor scaling, Horn's problem, Brascamp-Lieb constants, …

Appendix

The oracle

5.2 Gradient Methods

In this subsection, we study a generalization of gradient descent for solving the h-convex optimization problem (7). On the Euclidean space $M = \mathbb{R}^n$, given step sizes $s_k > 0$, the gradient descent updates iterates as $x_{k+1} = x_k - s_k \nabla f(x_k)$, or equivalently:[8]

$$x_{k+1} = \operatorname{argmin}_{x \in \mathbb{R}^n} \left\{ \frac{1}{2s_k} \|x_k - x\|^2 + \langle \nabla f(x_k), x - x_k \rangle \right\}.$$

We generalize this algorithm to arbitrary Hadamard manifolds M by replacing the squared norm with the squared distance on manifolds, and replacing the linear function $\langle \nabla f_i(x_k), x - x_k \rangle$ with the scaled Busemann function $B_{\nabla f_i(x_k)}(\overrightarrow{x_k x})$:

$$x_{k+1} = \operatorname{argmin}_{x \in M} \left\{ \frac{1}{2s_k} d(x_k, x)^2 + \frac{1}{m} \sum_{i=1}^{m} B_{\nabla f_i(x_k)}(\overrightarrow{x_k x}) \right\}. \tag{8}$$

When the objective function is a single h-convex function ($m = 1$), this algorithm simplifies to $x_{k+1} = \exp_{x_k}(-s_k \nabla f(x_k))$, a well-known form of the Riemannian gradient descent. While the oracle complexity of gradient descent for minimizing g-convex functions was studied to depend on the lower bound of sectional curvature (Zhang and Sra, 2016), we show here that curvature-free convergence rates are achievable when the objective function is the sum of h-convex functions.

Nesterov Accelerated Gradient Method

Non-strongly convex case. When $\mu = 0$, the accelerated gradient method updates the iterates as follows:

$$
\begin{aligned}
y_k &= x_k + \frac{2}{k+1}(z_k - x_k) \\
x_{k+1} &= y_k - \frac{1}{L}\nabla f(y_k) \\
z_{k+1} &= z_k - \frac{k+1}{2L}\nabla f(y_k),
\end{aligned} \tag{11}
$$

starting at an initial point $x_0 = z_0$. To solve the h-convex optimization problem (7), we propose the following generalization of this algorithm:

$$
\begin{aligned}
y_k &= \exp_{x_k}\left(\frac{2}{k+1}\log_{x_k}(z_k)\right) \\
x_{k+1} &= \exp_{y_k}\left(-\frac{1}{L}\nabla f(y_k)\right) \\
z_{k+1} &= \operatorname{argmin}_{z\in M}\left\{\frac{1}{2}d(z, z_k)^2 + \frac{k+1}{2L}\frac{1}{m}\sum_{i=1}^{m} B_{y_k,\nabla f_i(y_k)}(z)\right\}
\end{aligned} \tag{12}
$$

In particular, when $m = 1$, the updating rule for z_k simplifies to

$$
z_{k+1} = \exp_{z_k}\left(-\frac{k+1}{2L}\nabla B_{y_k,\nabla f(y_k)}(z_k)\right).
$$

Theorem 18 *The algorithm* (12) *satisfies*

$$
f(x_N) - f(x^*) \leq \frac{2L}{N^2}d(x_0, x^*)^2.
$$

Smooth horo-convex interpolation

Claim 1: Consider the data $D = \{(x_i, g_i, f_i)\}_{i=1,\ldots,n}$, and define the data

$$\tilde{D} = \left\{(\tilde{x}_i, \tilde{g}_i, \tilde{f}_i) := \big(\exp_{x_i}(-\frac{g_i}{L}), \Gamma_{x_i, -\frac{g_i}{L}} g_i, f_i - \frac{\|g_i\|^2}{2L}\big)\right\}_{i=1,\ldots,n}.$$

If $\tilde{D}$ is interpolated by a g-convex function $\tilde{f}$ then D is interpolated by the g-convex function $f(x) = \min_{y\in M}\{\tilde{f}(y) + \frac{L}{2}d(x,y)^2\}$ [a Moreau envelope!]. f is L-B-smooth. If $\tilde{f}$ is B-convex then f is B-convex.