Curvature and Complexity:

Better lower bounds for geodesically convex optimization

COLT 2023

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Setting: g-convex optimization

$$\min_{x \in \mathcal{M}} f(x) \qquad \textbf{(P)}$$

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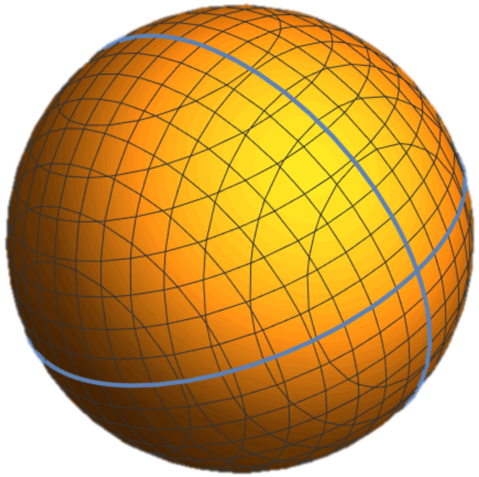

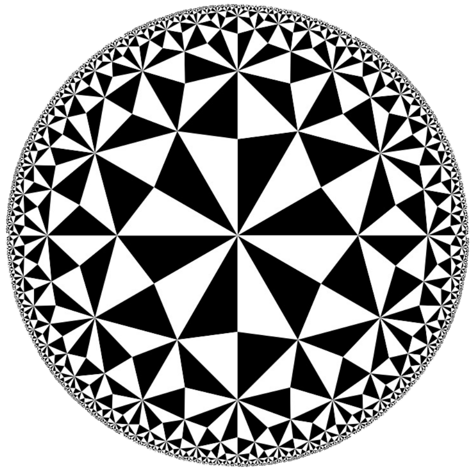

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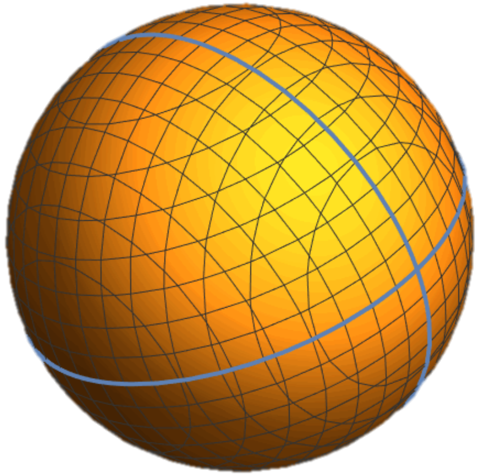

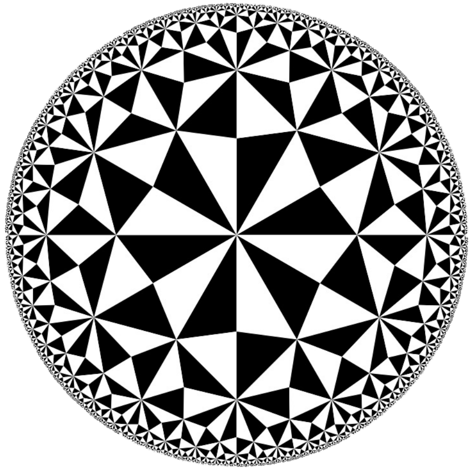

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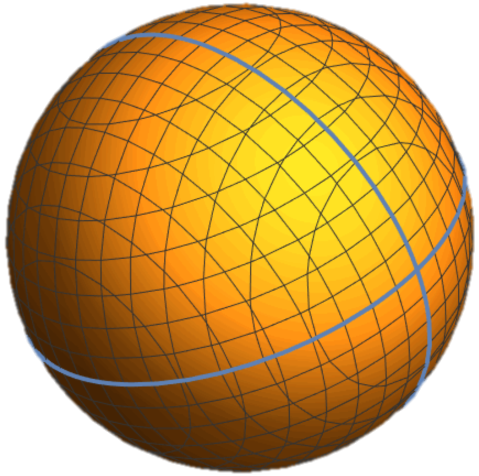

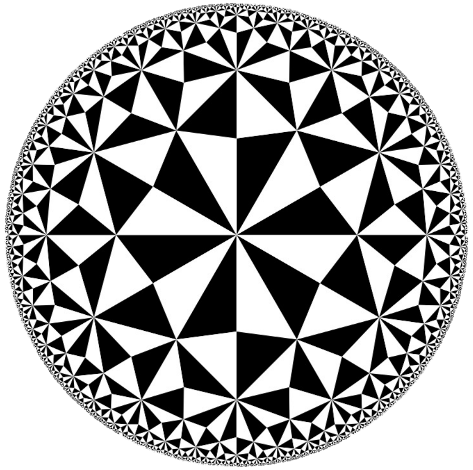

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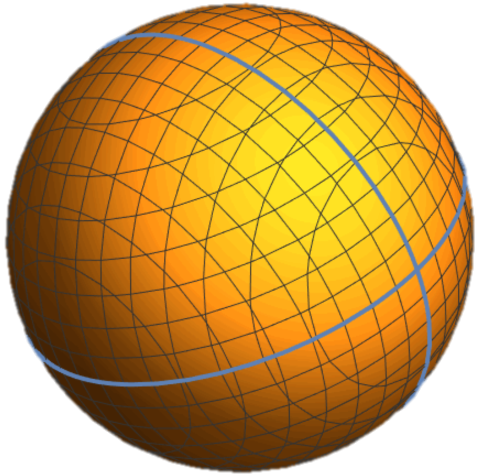

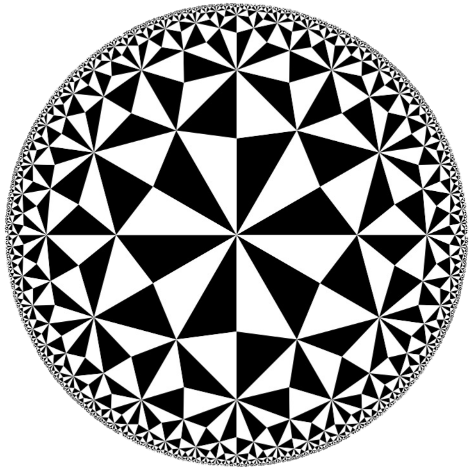

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$\min_{x \in \mathcal{M}} f(x)$ **(P)**

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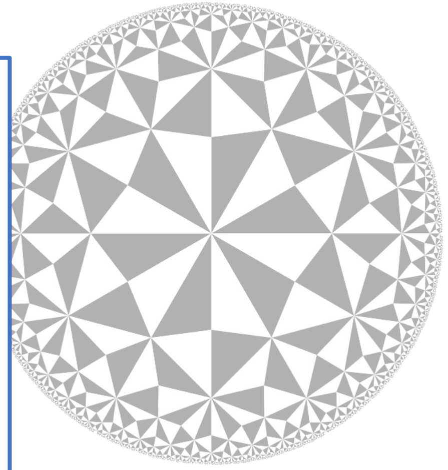

f is *geodesic*

Def: For every

function

Question: co $\mathcal{M}$?

Most positively curved spaces do not carry nonconstant global g-convex functions.

Applications of g-convexity usually have K $\leq$ 0:

- *Operator scaling*
- *Robust covariance estimation*
- *Matrix normal models*
- *Intrinsic medians (e.g., for phylogenetics)*

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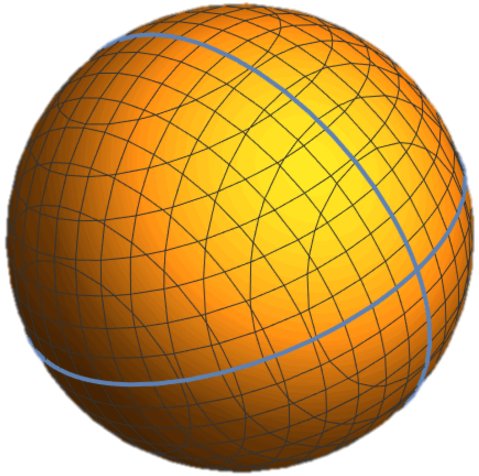

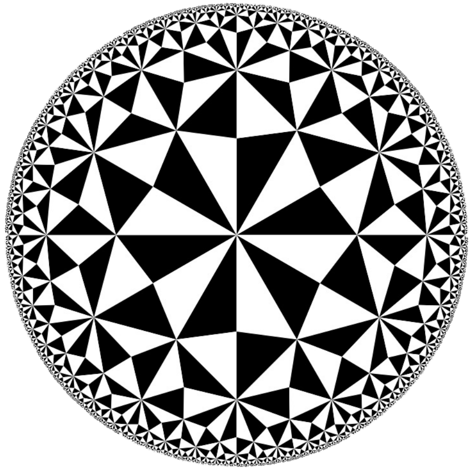

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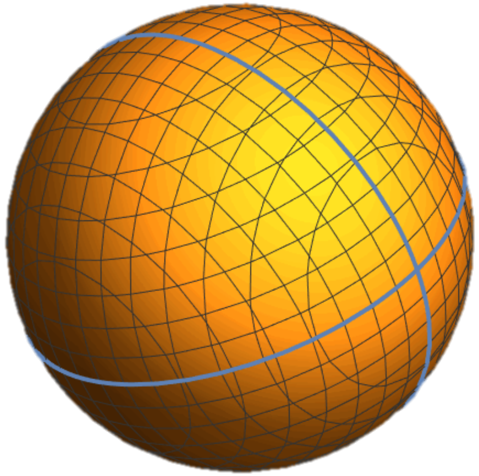

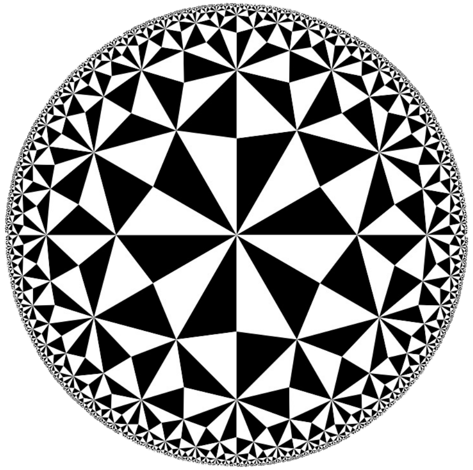

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Assume $\mathcal{M}$ has constant curvature (simplifying assumption)

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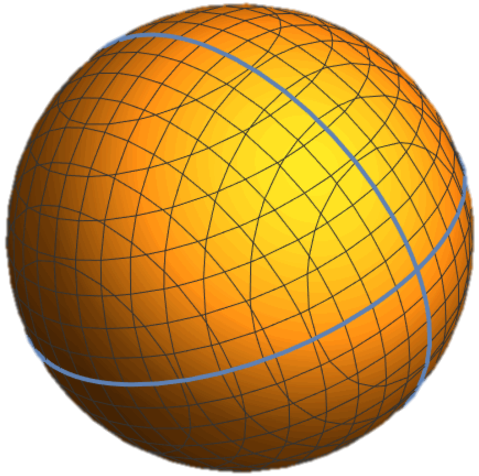

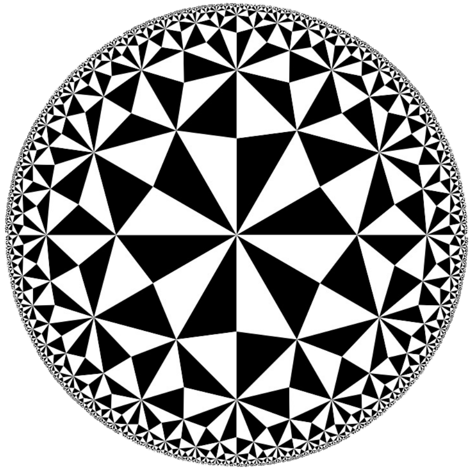

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Throughout $\mathcal{M} = \mathbb{H}^d$ is a hyperbolic space of curvature $K = -1$ (simplifying assumption)

Curvature

$K = 0$

$K < 0$

Constant curvature:

Euclidean space $\mathbb{R}^d$

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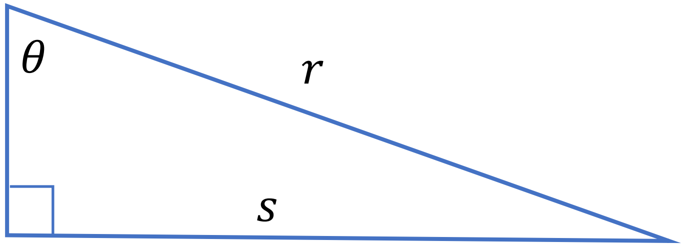
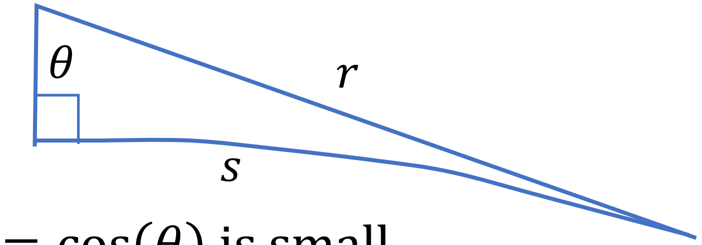
Pythagorean thm:		

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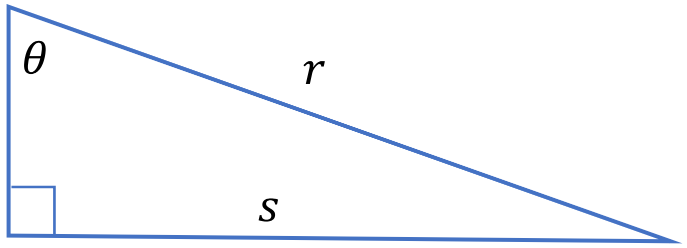

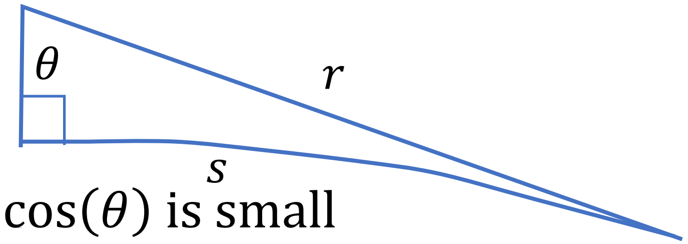

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Pythagorean thm:

$s^2 = (1 - \epsilon^2)r^2$

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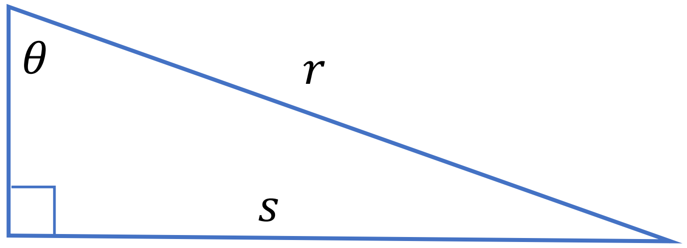

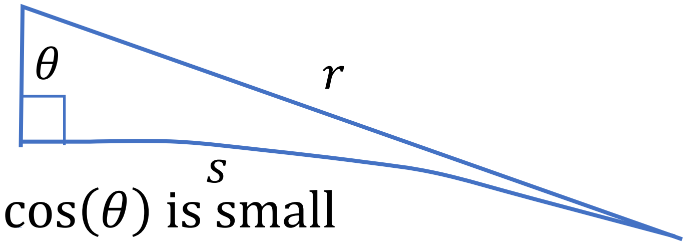

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$$\zeta = \frac{r\sqrt{|K|}}{\tanh(r\sqrt{|K|})} \sim 1 + r\sqrt{|K|}$$

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- (P1) Lipschitz & Low-dimensional (d fixed)
- (P2) Lipschitz $|f(x) - f(y)| \le M \operatorname{dist}(x, y)$
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Worst-case complexity for (P2) & (P3): can depend on both ϵ and $\zeta \sim 1 + r\sqrt{|K|}$

Main results

$\zeta \sim 1 + r\sqrt{|K|}$

g-convex setting	Lower bound	Upper bound	Algorithm
(P1) Lipschitz, lo-dim		$O(\zeta d^2)$	Centerpoint method (Rusciano'19)
(P2) Lipschitz		$O\left(\frac{\zeta}{\epsilon^2}\right)$	Subgradient descent (Zhang & Sra'16)
(P3) Smooth		$\tilde{O}\left(\sqrt{\zeta/\epsilon}\right)$	RNAG-C (Kim & Yang'22 / Martinez-Rubio et al.'22)
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Curvature dependence in all upper bounds is unavoidable!

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Slight modification of hard function in C & Boumal'22

Volume of ball of radius r: Vol $\sim e^{dr}$

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Do not match!

Note: $\kappa \geq \zeta$ and $\frac{1}{\epsilon} \geq \zeta$

Additive

Multiplicative

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Cutting... game			...method (Rusciano'19)

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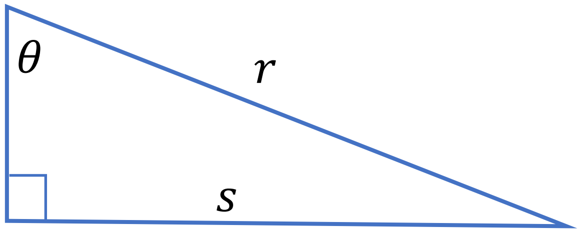
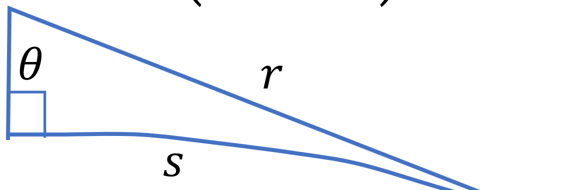


$$s^2 \approx \left(1 - \frac{\epsilon^2}{\zeta}\right) r^2$$



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Choosing halfspaces in right way yields lower bound $\widetilde{\Omega}\left(\frac{1}{\zeta^2 \epsilon^2}\right)$.

Smooth lower bound (P3)

Worst function in world: $f(x) = x_{(1)}^2 - 2x_{(1)} + \sum_i \left(x_{(i)} - x_{(i+1)}\right)^2 + x_{(k)}^2$

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Smooth the Lipschitz lower bound with Moreau envelope (Guzman and Nemirovski'15):

$$f_\lambda(x) = \inf_{y \in \mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} \mathrm{dist}^2(x, y) \right\}$$

Smooth lower bound (P3)

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Surprising because no good notion of Fenchel dual on manifolds.

Yields lower bound $\tilde{\Omega}\left(\frac{1}{\zeta\sqrt{\epsilon}}\right)$.

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L_i are cleverly chosen hyperbolic halfspaces.

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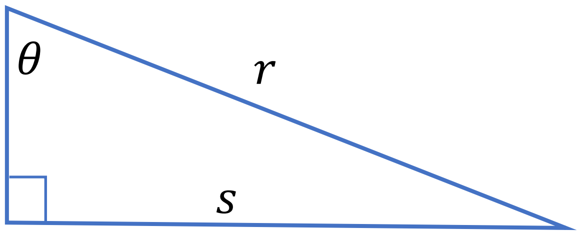
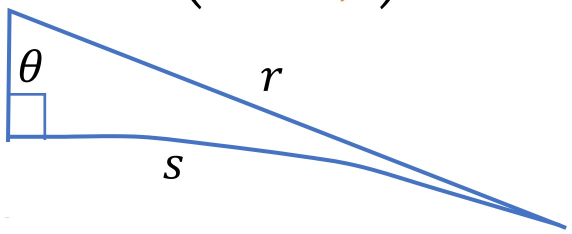
$$s^2 = (1 - \epsilon^2) r^2$$



$$s^2 \approx \left(1 - \frac{\epsilon^2}{\zeta}\right) r^2$$



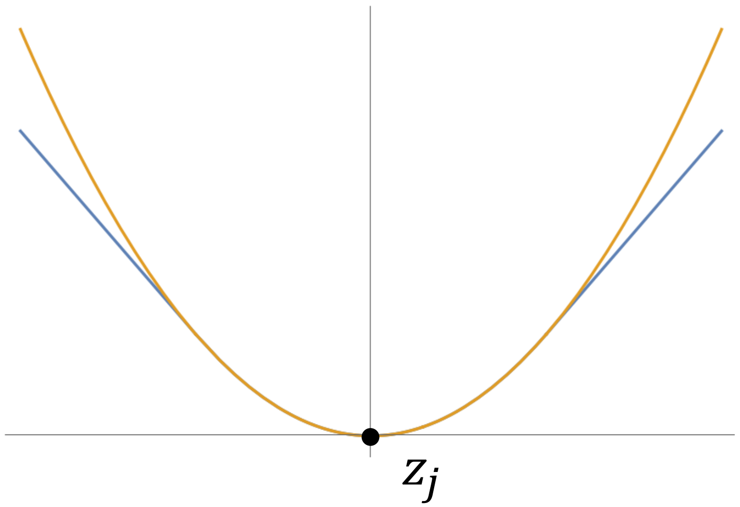
$\epsilon = \cos(\theta)$ is small

Appendix

Sec 3

Slight modification of hard function in C & Boumal'21

Volume of ball of radius r: Vol $\sim e^{dr}$



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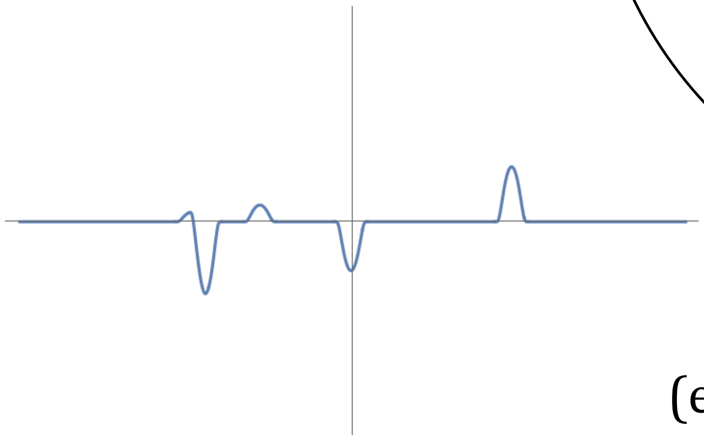
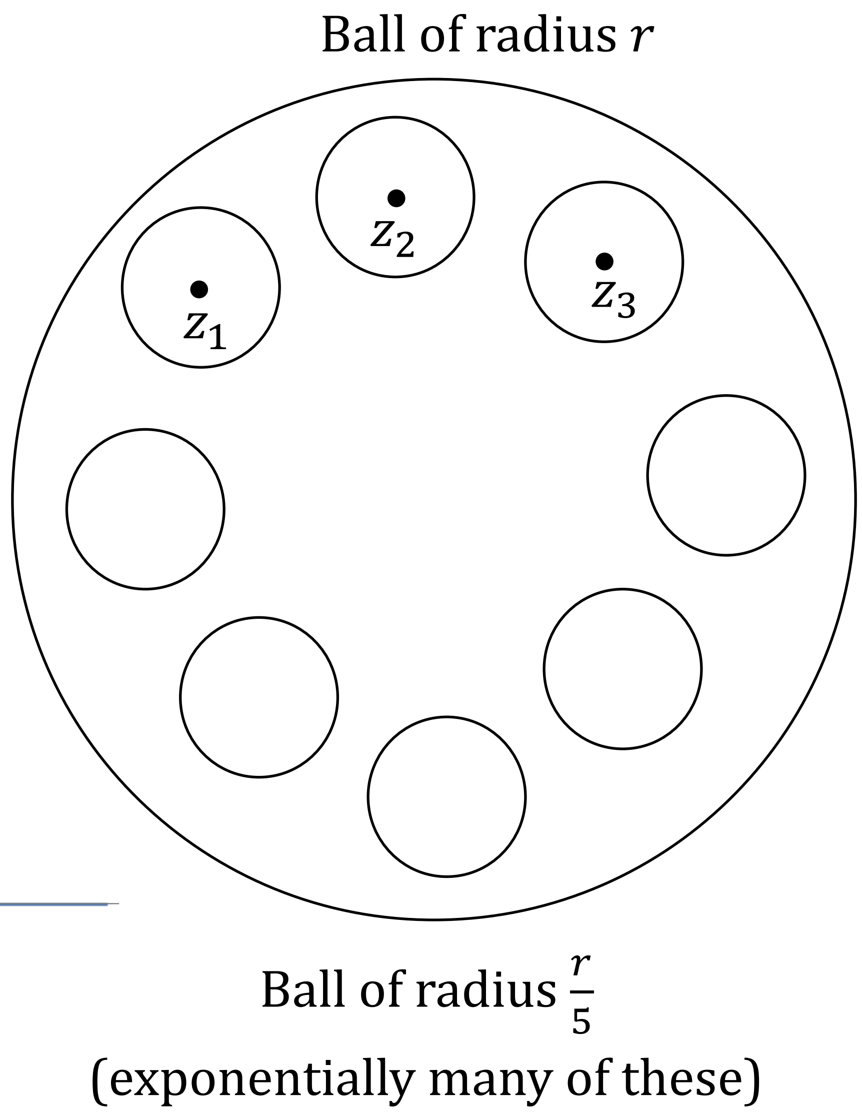




Lipschitz lower bound (P2) – Sec 4

In Euclidean space: $f(x) = \max_{i=1,\ldots,d} \{\langle s_i e_i, x \rangle\}, s_i \in \{-1, +1\}$

Fundamental difficulty: No good notion of linear functions on manifolds.

- Every convex function equals a sup of affine functions, which are themselves convex.
- Every g-convex function equals a sup of $x \mapsto F_i + \langle g_i, \log_{y_i}(x) \rangle$, but these are not g-convex.

Totally geodesic submanifolds: $S_i = \{x \in \mathcal{M} = \mathbb{H}^d : \langle g_i, \log_{y_i}(x) \rangle = 0\}$

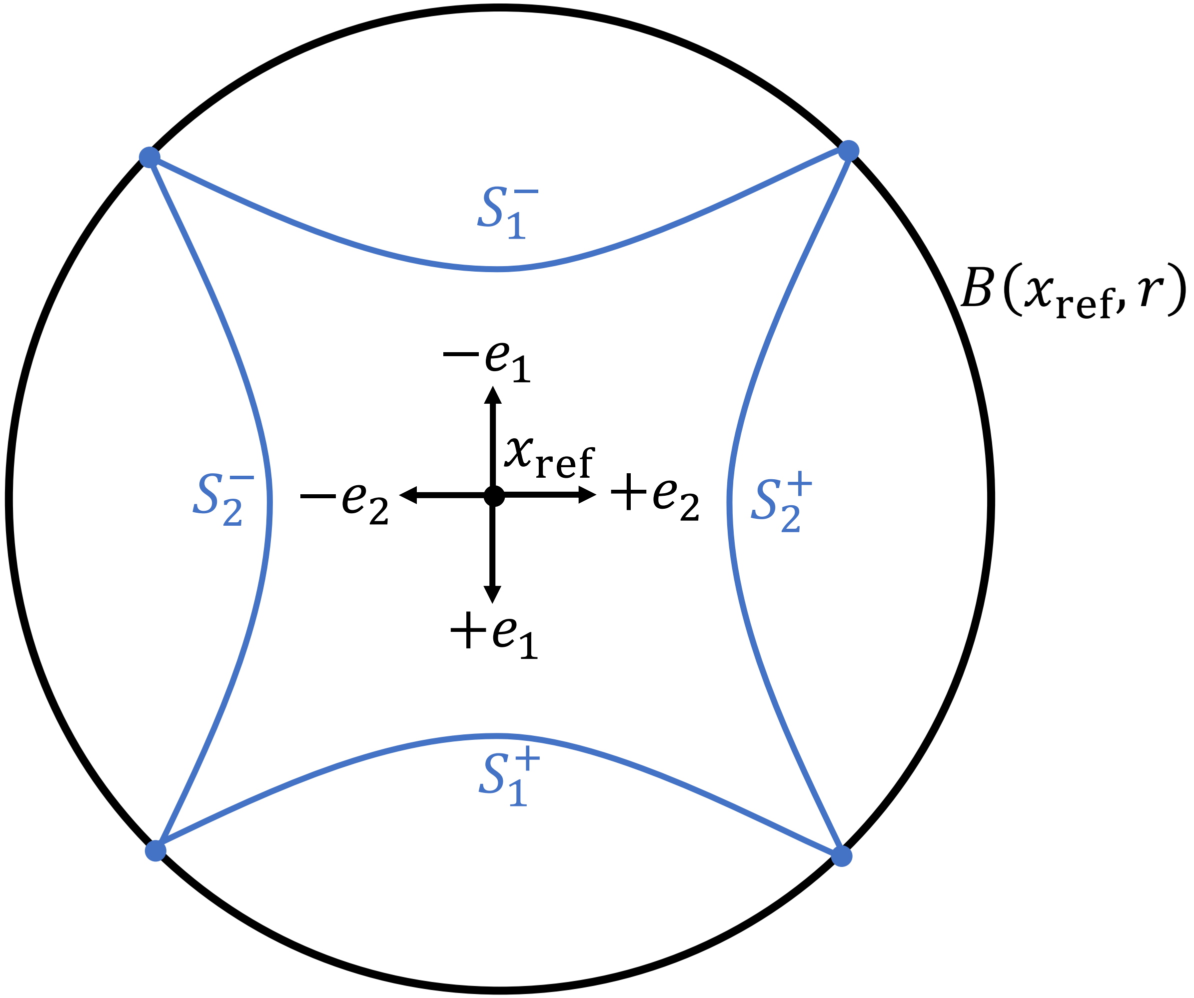
- Generalization of affine subspace

Idea: use max of distance to totally geodesic submanifolds:

$$f(x) = \max_{i=1,\ldots,d} \{\mathrm{dist}(x, S_i^{s_i})\}, s_i \in \{-1, +1\}$$

Choosing halfspaces in right way yields lower bound $\tilde{\Omega}\left(\frac{1}{\zeta^2 \epsilon^2}\right)$.

Lipschitz lower bound (P2) – Sec 4



Smooth lower bound (P3) – Sec 5

Worst function in world: $f(x) = x_{(1)}^2 - 2x_{(1)} + \sum_i (x_{(i)} - x_{(i+1)})^2 + x_{(k)}^2$

Smooth Lipschitz lower bound with Moreau envelope (Guzman and Nemirovski'15):

$$f_\lambda(x) = \inf_{y \in \mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} \mathrm{dist}^2(x, y) \right\}$$

Surprising because no good notion of Fenchel dual on manifolds.

Yields lower bound $\widetilde{\Omega}\left(\frac{1}{\zeta\sqrt{\epsilon}}\right)$.

Tight lower bound for subgradient descent (P2) – Sec 6

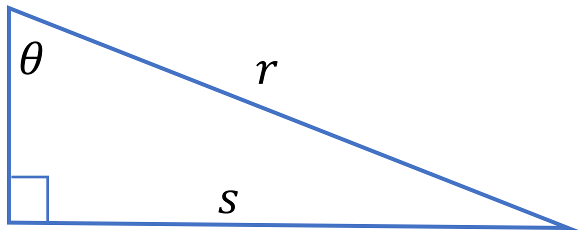
New worst function in the world (*not* an extension of a Euclidean proof):

$$f(x) = \mathrm{dist}(x, x^*) + \max_{i=0,\dots,d-2} \left\{ \frac{1}{4\epsilon} \mathrm{dist}(x, L_i) \right\}$$

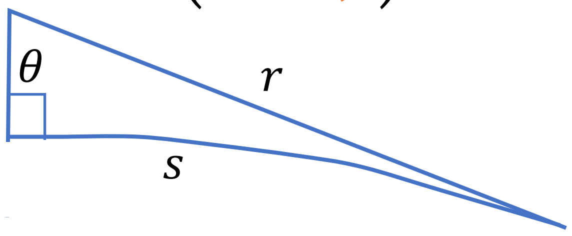
L_i are cleverly chosen hyperbolic halfspaces.

Yields $\Omega\left(\frac{\zeta}{\epsilon^2}\right)$ lower bound for subgradient descent (with Polyak step size).

$$s^2 = (1 - \epsilon^2) r^2$$



$$s^2 \approx \left(1 - \frac{\epsilon^2}{\zeta}\right) r^2$$



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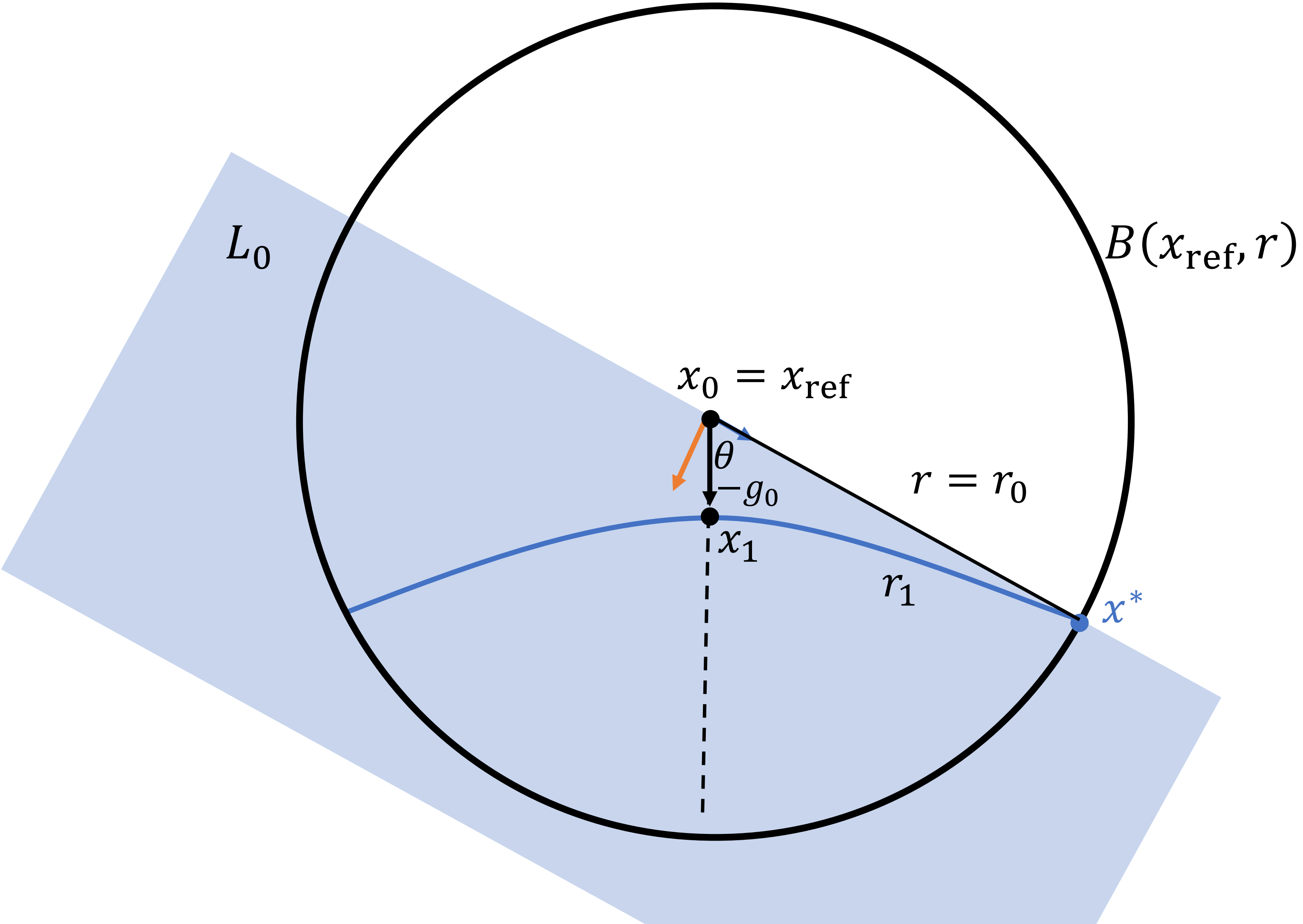
Tight lower bound for subgradient descent (P2) – Sec 6



Cutting-planes game (proxy for (P1)) – Sec 7

Cutting planes game:

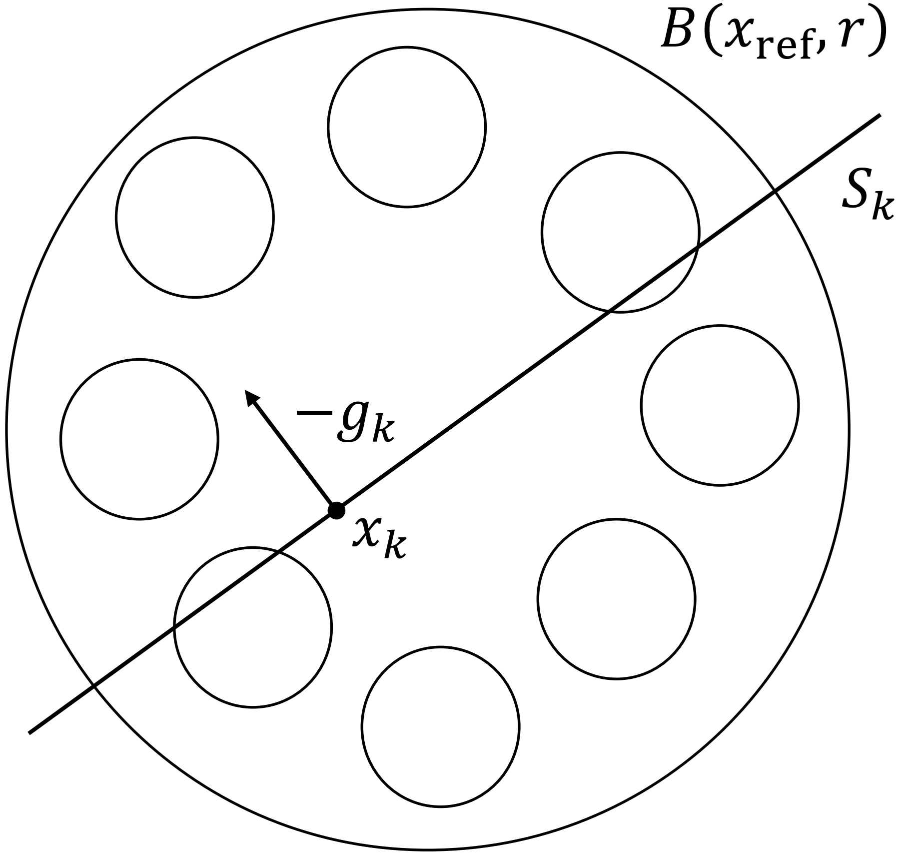
- Initially given $B(x_{\text{ref}}, r)$ containing x^*
- Upon query x_k, oracle returns tangent vector g_k such that $\langle g_k, \log_{x_k}(x^*) \rangle \le 0$
- Task: Find x_k so that $B(x^*, \epsilon r)$ intersects $S_k = \{x \in \mathcal{M} : \langle g_k, \log_{x_k}(x) \rangle = 0\}$

Usual lower bounds proof (tessellation of cube) does not generalize (Nesterov'04).

Width-bounded separators (Kisfaludi-Back'20, Fu'11):

- Given a collect of balls and a point x_k, there exists a totally geodesic submanifold S_k through x_k which intersects a small fraction of these balls

Yields $\tilde{\Omega}(\zeta d)$ lower bound for cutting-plane schemes, e.g., center-of-gravity method.



G-convex interpolation – Sec 8

Interpolation is crucial for building lower bounds (Taylor et al.'16).

A collection of function values and tangent vectors $(F_i, x_i, g_i)_{i=1}^N$ is interpolated by a g-convex function f if $f(x_i) = F_i$ and $g_i \in \partial f(x_i)$ for all i.

A convex function interpolates $(F_i, x_i, g_i)_{i=1}^N$ if and only if

$$F_j \geq F_i + \langle g_i, x_j - x_i \rangle \text{ for all } i, j$$

For g-convex functions the analogous naïve necessary conditions are *not* sufficient for interpolation even for just 3 points:

- There exists $(F_i, x_i, g_i)_{i=1}^3$ such that $F_j \geq F_i + \left\langle g_i, \log_{x_j}(x_j) \right\rangle$ for all i, j, yet this data cannot be interpolated by a g-convex function.