Negative curvature obstructs acceleration for g-convex optimization,
even with exact first-order oracles

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Short answer: No.

Slightly longer answer: We show there are Riemannian manifolds and regimes where gradient descent is optimal (worst-case complexity).

Builds on work of Hamilton and Moitra (2021), who show the answer is no when algorithms receive noisy information.

Hamilton and Moitra: "A No-Go Theorem for Acceleration in the Hyperbolic Plane" (2021)

Geodesically convex optimization

$$\min_{x \in D} f(x)$$

Search space D is a g-convex subset of a Riemannian manifold $\mathcal{M}$:

Cost f is μ-strongly g-convex:

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For each $x, y \in D$, there is a unique minimizing geodesic $t \mapsto \gamma(t)$ contained in D, connecting x, y.

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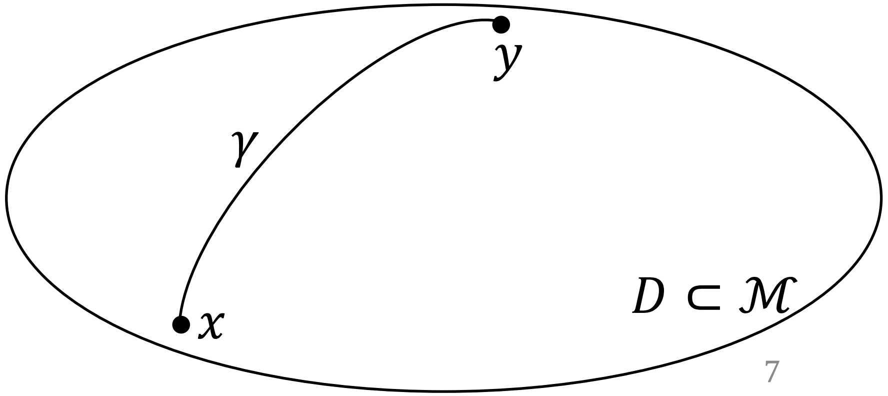

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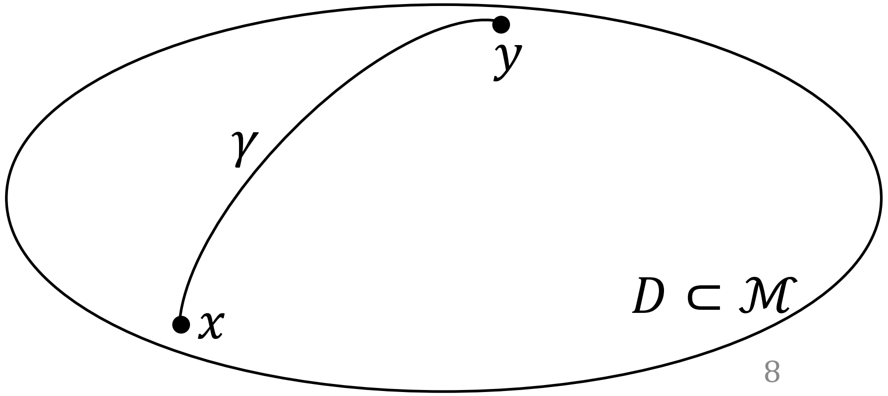

Cost f is μ-strongly g-convex:

$$t \mapsto f\big(\gamma(t)\big)$$

is μ-strongly convex for any geodesic γ in D.

Hadamard manifolds

Complete, simply connected, with non-positive (intrinsic) curvature.

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Positive definite matrices: $\mathcal{M} = \{P \in \mathbf{R}^{n \times n} : P = P^\top \text{ and } P \succ 0\}$
with affine-invariant metric $\langle X, Y \rangle_P = \mathrm{Tr}(P^{-1} X P^{-1} Y)$.

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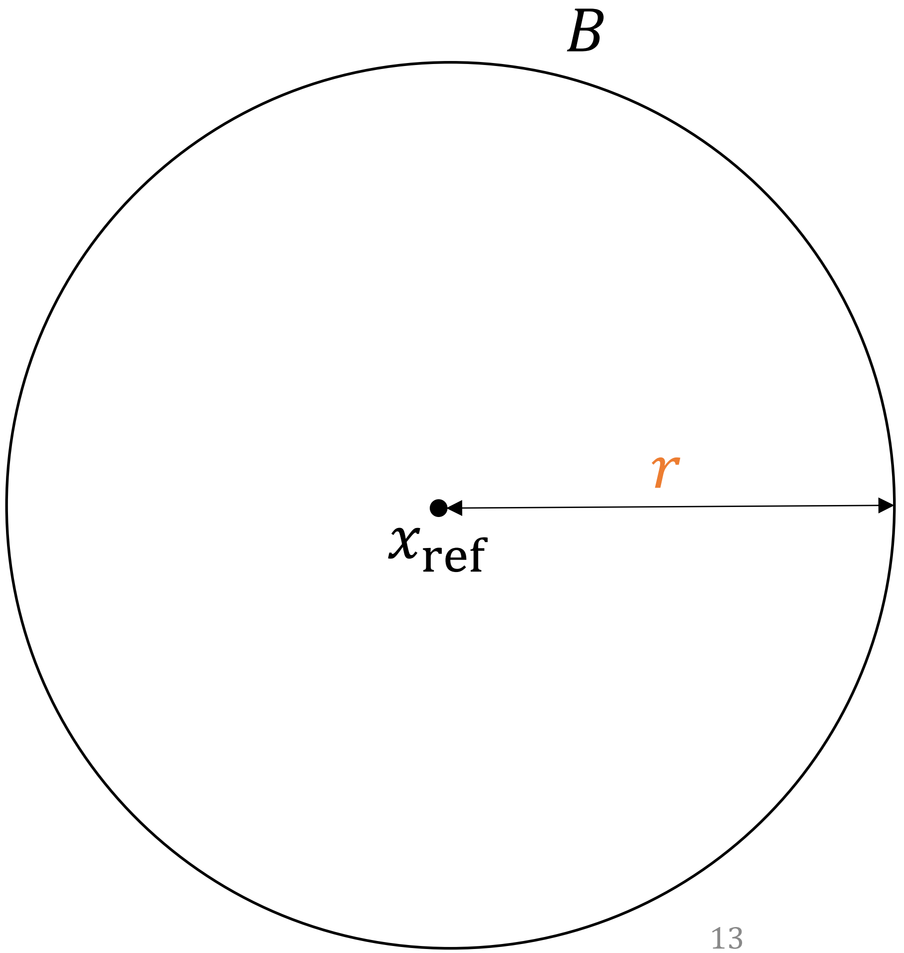
Non-example: Sphere

Computational task

Geodesic ball $B = B(x_{\mathrm{ref}}, r)$ of radius r in Hadamard space $\mathcal{M}$.



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Geodesic ball $B = B(x_{\mathrm{ref}}, r)$ of radius r in Hadamard space $\mathcal{M}$.

You know:

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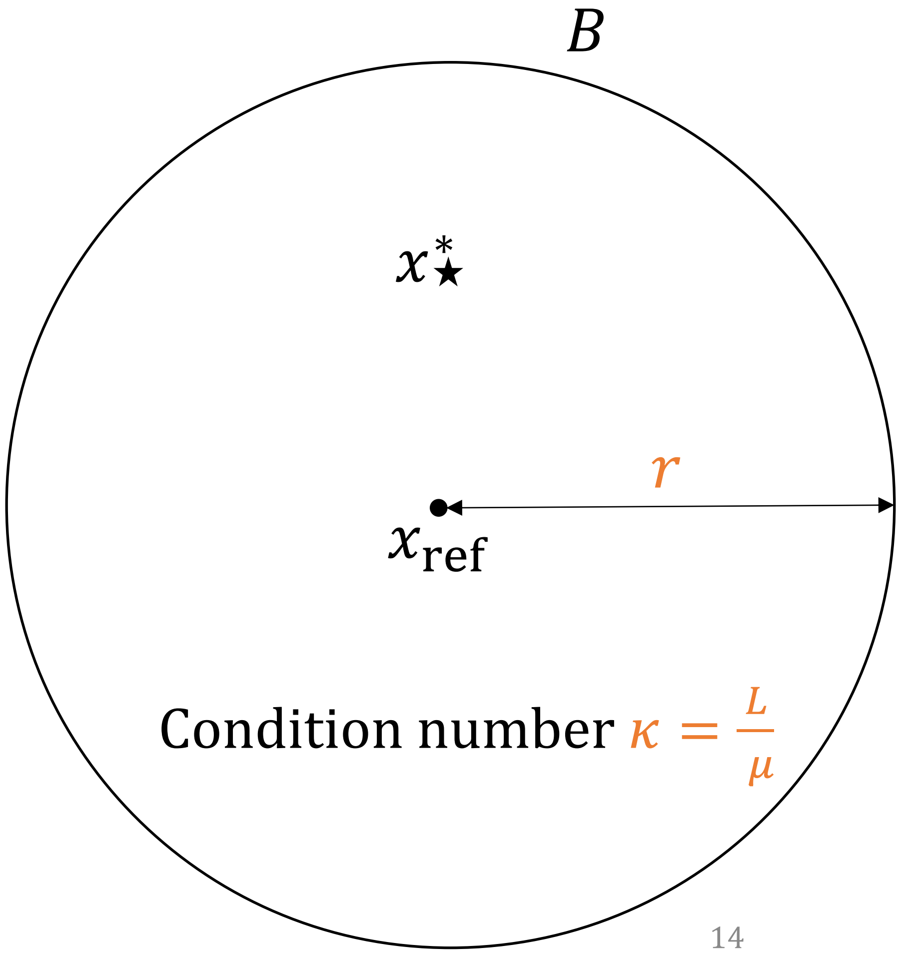

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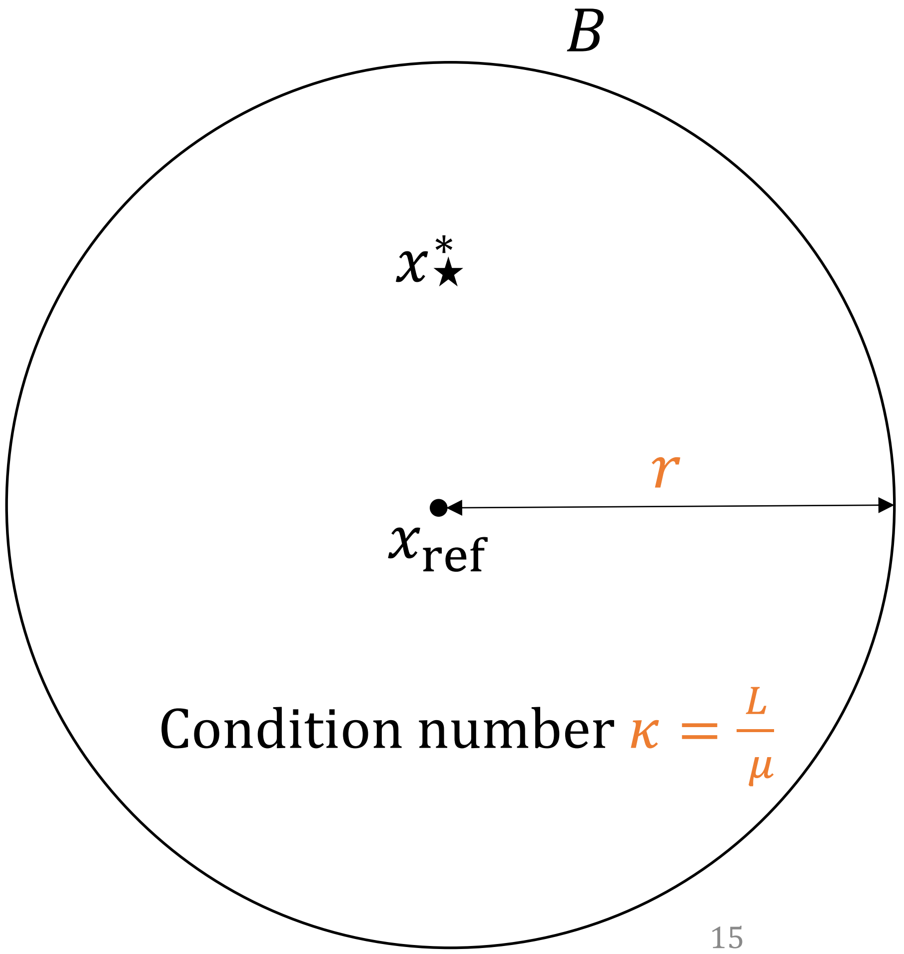

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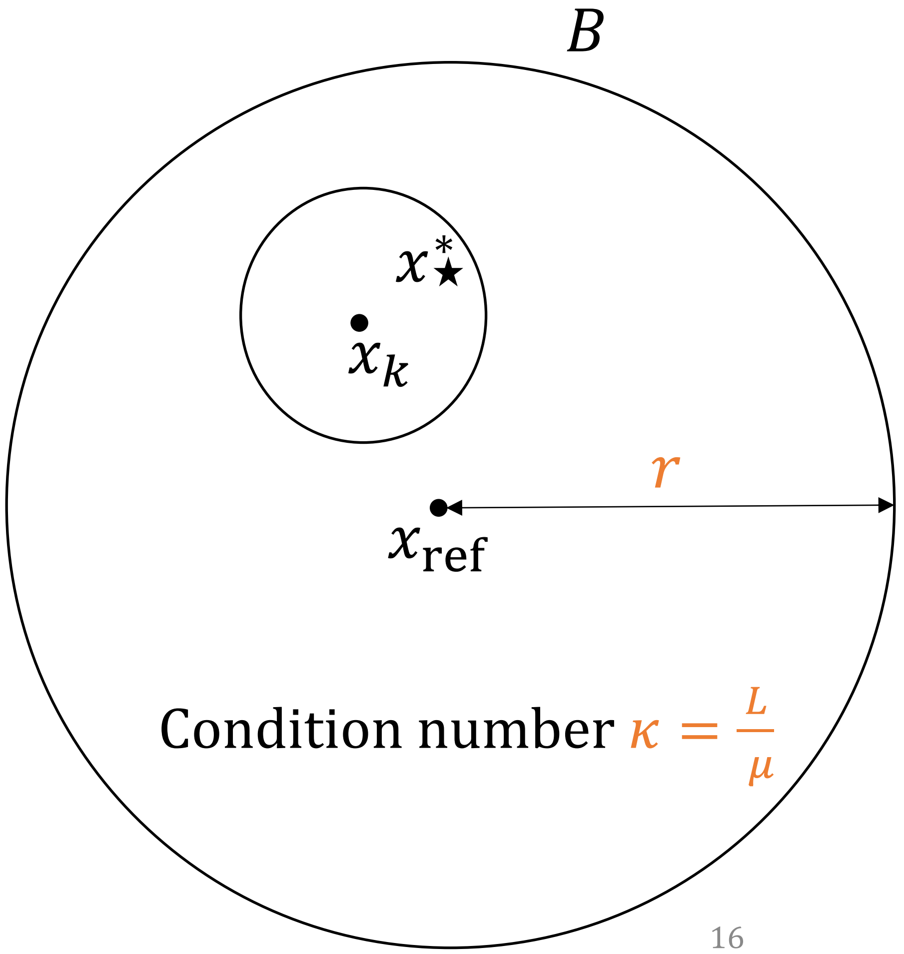
Task: find a ball of radius $r/5$ containing x^*.

Least number of oracle queries necessary?



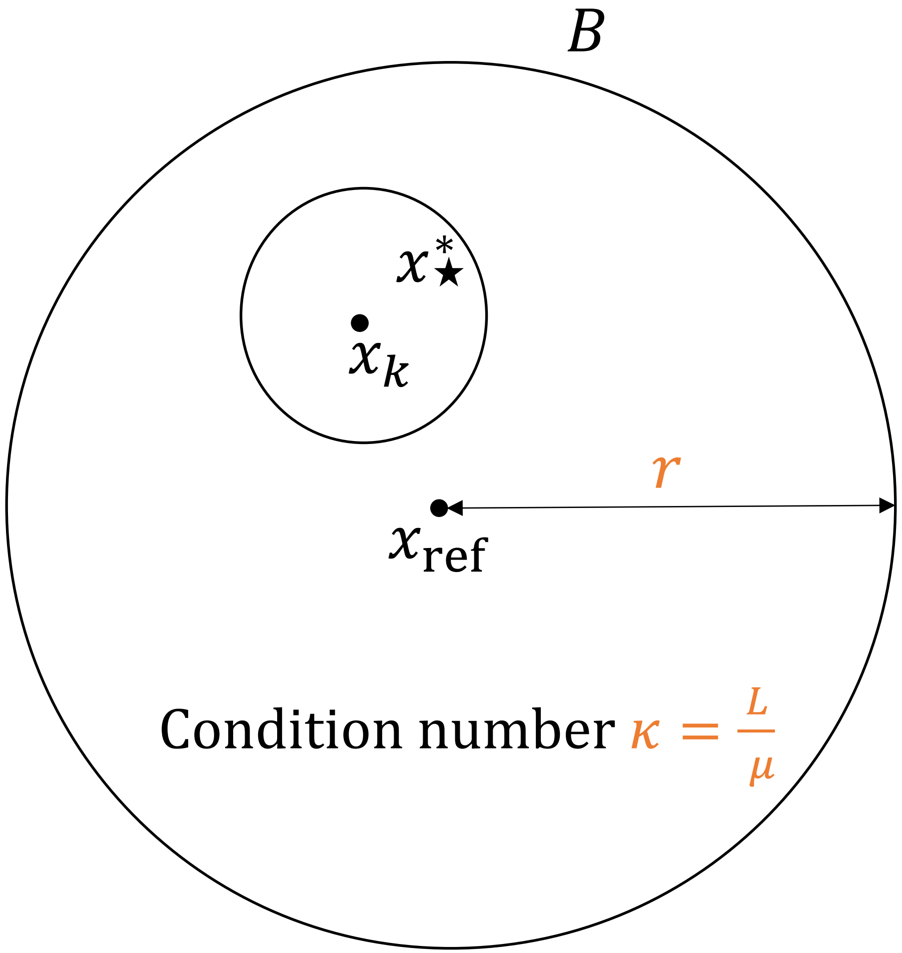
What happens in $\mathbb{R}^d$?

If $\mathcal{M} = \mathbb{R}^d$:

Gradient Descent (GD)

$$x_{k+1} = x_k - \eta \nabla f(x_k)$$

$O(\kappa)$ oracle queries.



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$$y_k = x_k + (1-\theta)v_k$$
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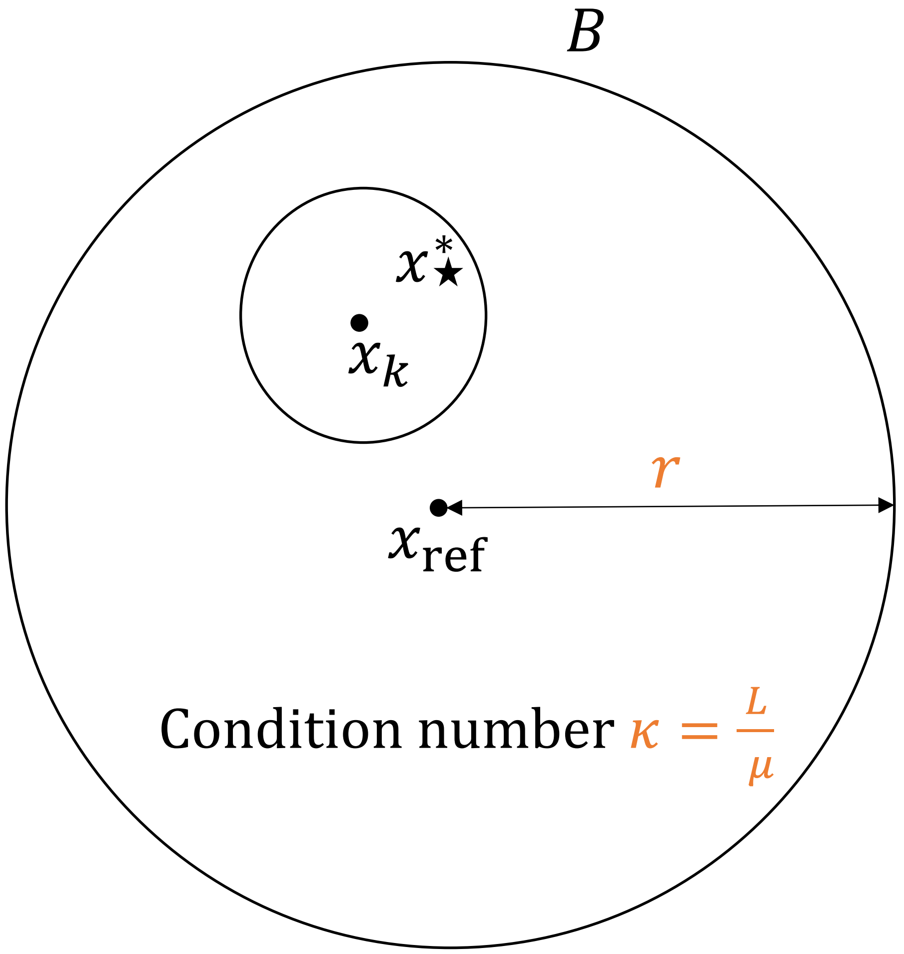
$$v_{k+1} = x_{k+1} - x_k$$

$\tilde{O}(\sqrt{\kappa})$ oracle queries.



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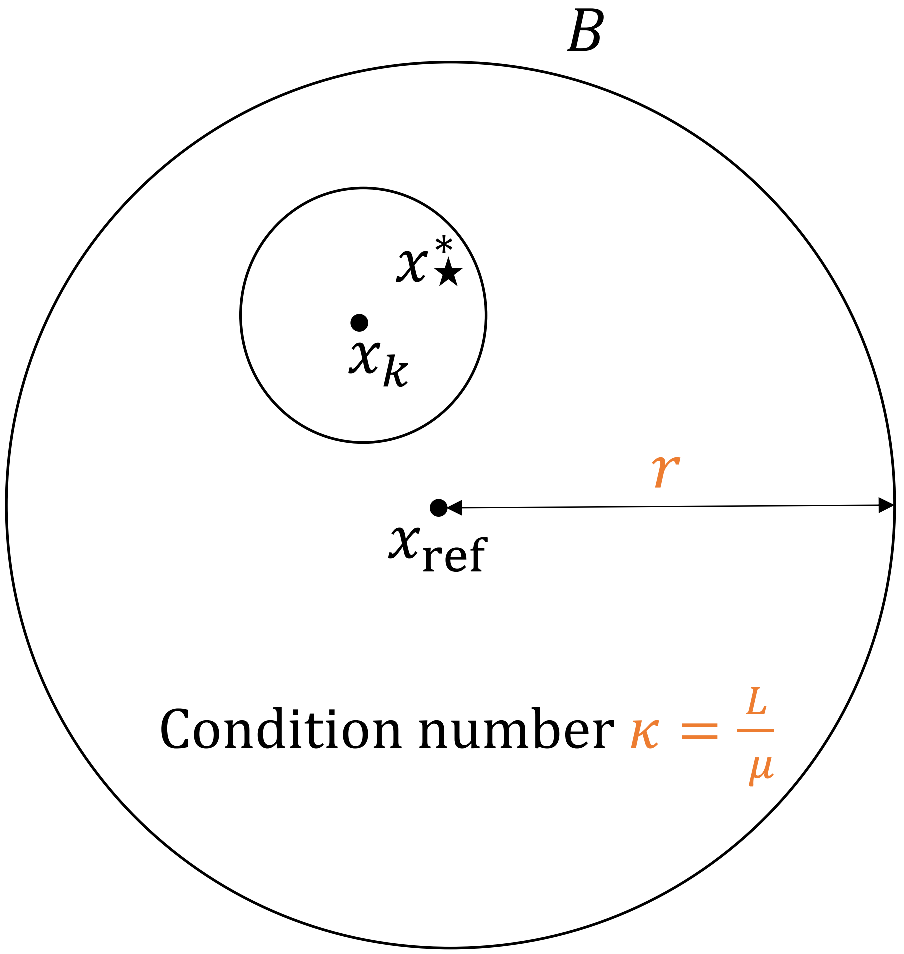
NAG has optimal oracle complexity; GD does not.



Optimal methods

What about on Riemannian manifolds?

Riemannian GD (RGD) requires $O(\kappa)$ oracle queries (when for example $\mathcal{M}$ is a hyperbolic space).

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Partial positive result (Zhang, Ahn, Sra, Martinez-Rubio, Alimisis, et al.): you can accelerate in some cases (e.g., r small).

Main results

Let $\mathcal{M}$ be a Hadamard manifold of dimension $d \geq 2$ whose sectional curvatures are in the interval $[\mathrm{K}_{lo}, \mathrm{K}_{up}]$ with $\mathrm{K}_{up} < 0$.

Let $r = c_2\, \kappa \,/\sqrt{-K_{lo}}$.

For hyperbolic spaces, $K_{lo} = K_{up} = K < 0$

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For every deterministic algorithm $\mathcal{A}$, there is a C^∞ function f which is

- 1-strongly g-convex in all of $\mathcal{M}$;
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$$\Omega\left(\sqrt{\frac{K_{up}}{K_{lo}}\frac{\kappa}{\log \kappa}}\right) \Longrightarrow$$

$O(\sqrt{\kappa})$ rate is impossible;
RGD is optimal (up to log).

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Other settings

$n \times n$ positive definite matrices with affine-invariant metric.

Smooth nonstrongly g-convex optimization ($\mu = 0$).

Nonsmooth g-convex optimization.

Negative curvature

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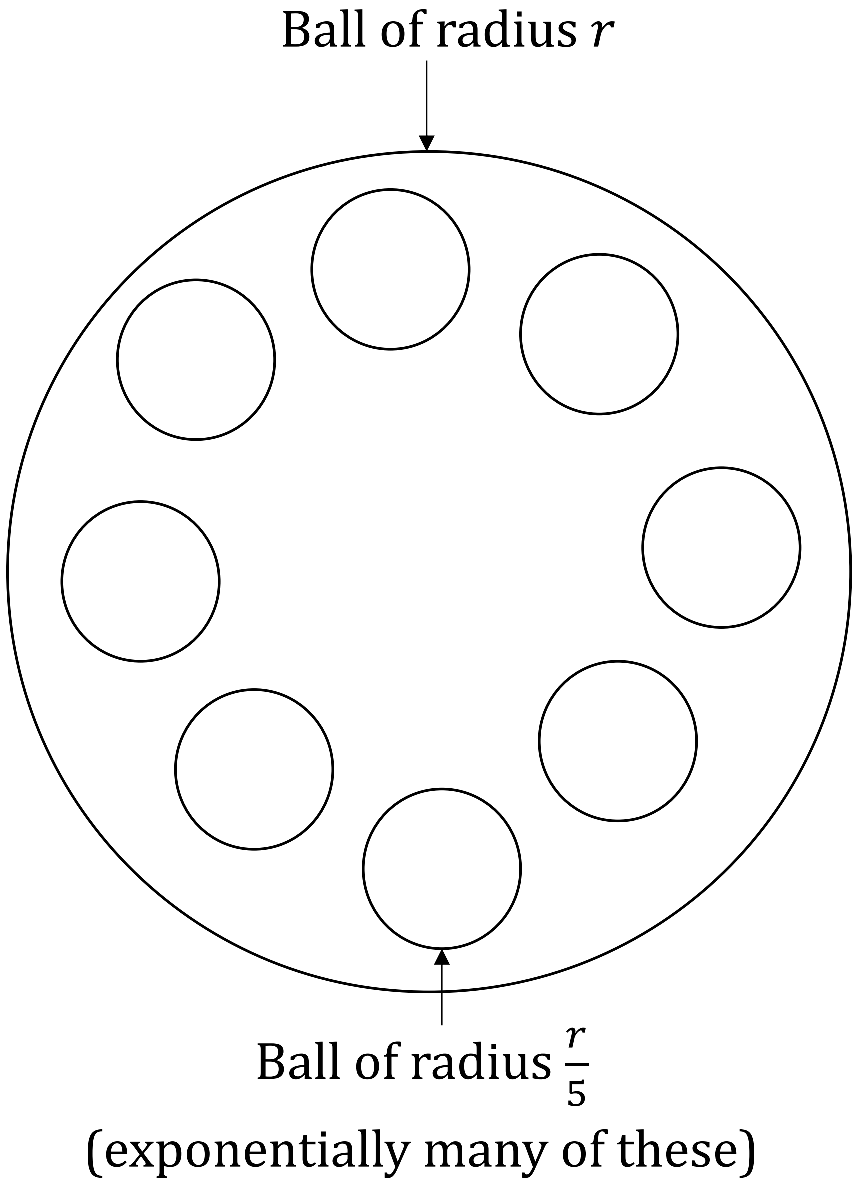
How many disjoint balls of radius $r/5$ contained in every ball of radius r?

$e^{\Theta(rd)}$ in hyperbolic space

$e^{\Theta(d)}$ in Euclidean space



Future directions

Tighter upper/lower bounds, e.g., Kim and Yang (2022)

Randomized algorithms which receive exact information?

Ellipsoid method?

Interior-point methods?

Proof technique

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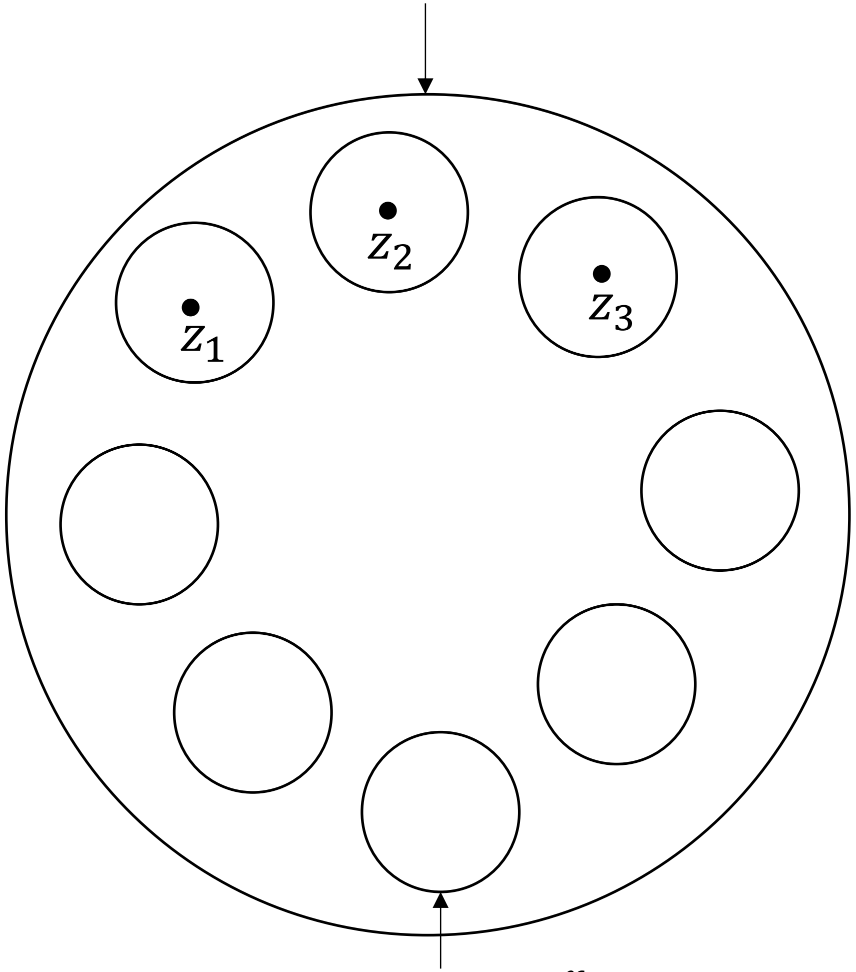
Ball of radius r



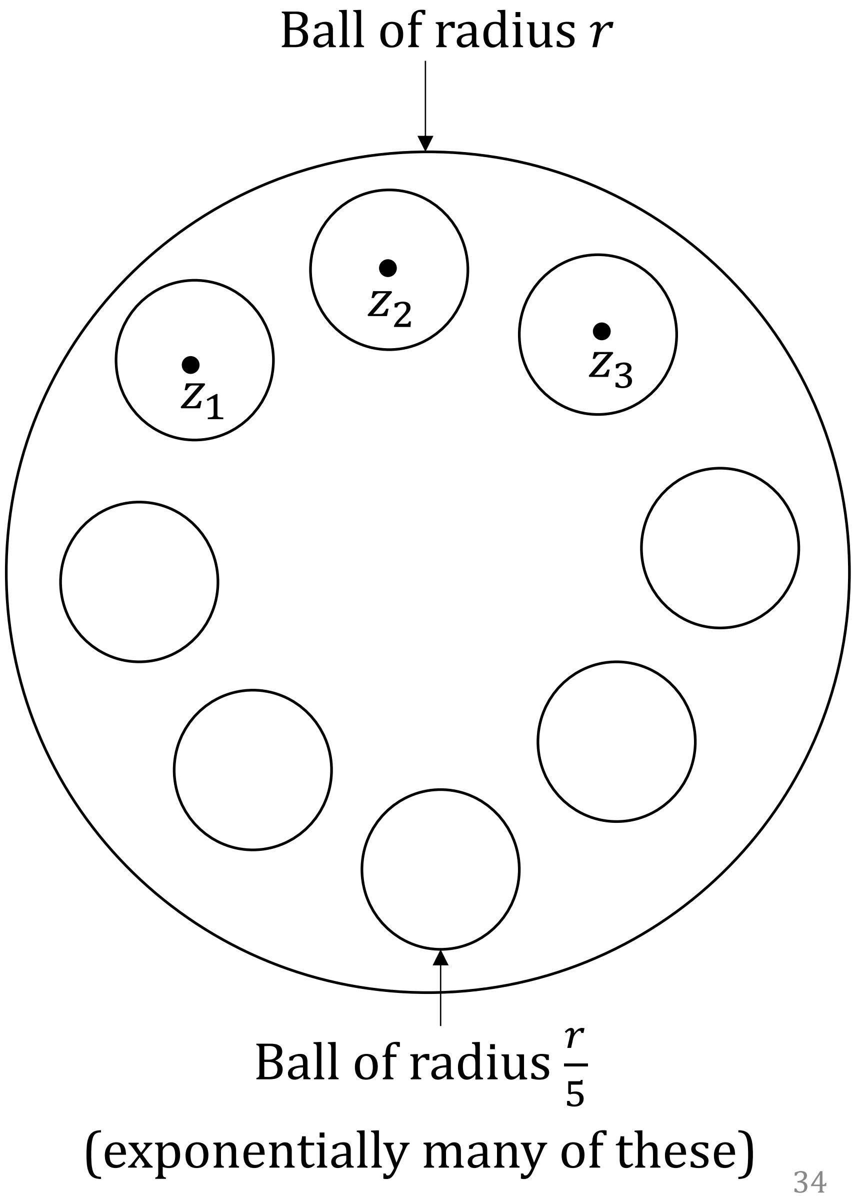
Ball of radius $\frac{r}{5}$
(exponentially many of these)

Proof technique

Hamilton and Moitra consider the functions

$$x \mapsto \frac{1}{2}\,\mathrm{dist}\left(x, z_j\right)^2, j = 1, \ldots, N$$

Show that in expectation (over noisiness of queries), any algorithm makes at most limited progress per query.



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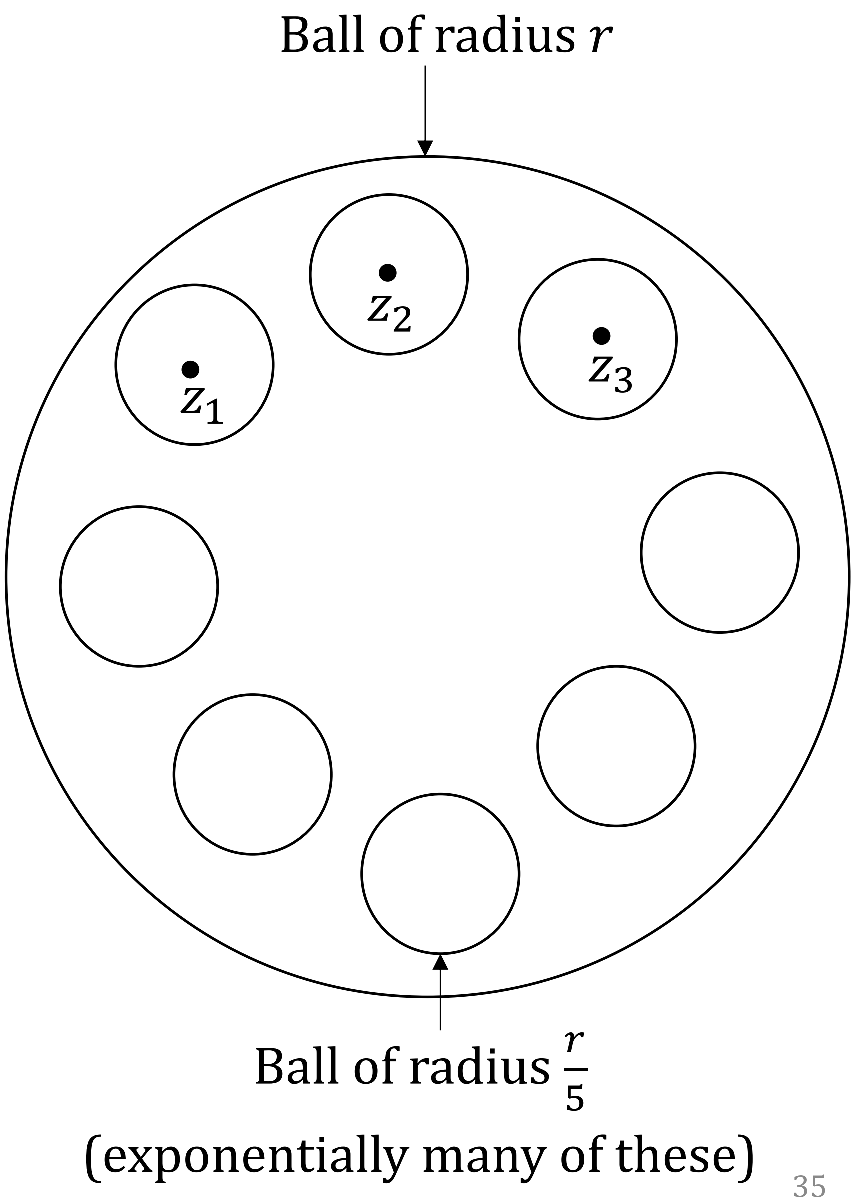
Gradients of these functions point directly towards the minimizer

- Ok if there is noise
- A problem if queries are exact



Proof technique

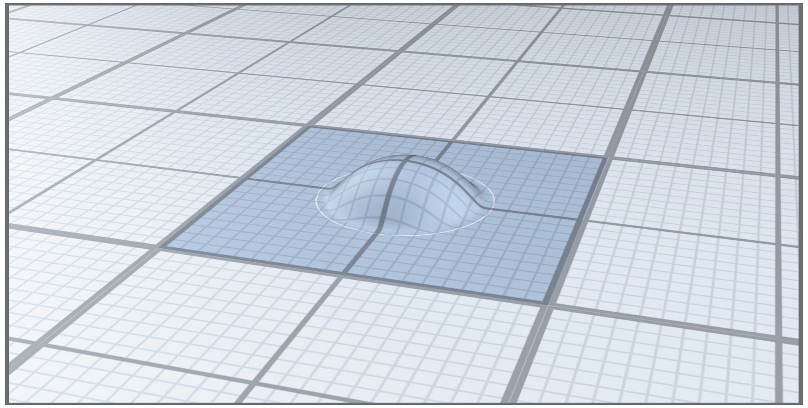
Our solution:

The hard functions we consider are squared distance functions plus a perturbation

$$x \mapsto \frac{1}{2}\operatorname{dist}(x, z_j)^2 + H_{j,k}(x), \qquad \|\operatorname{Hess} H_{j,k}(x)\| \leq \frac{1}{2}.$$

For any algorithm, the perturbation $H_{j,k}$ is constructed adversarially using a resisting oracle.

Proof technique



Our solution:

Perturbation is a sum of bump functions

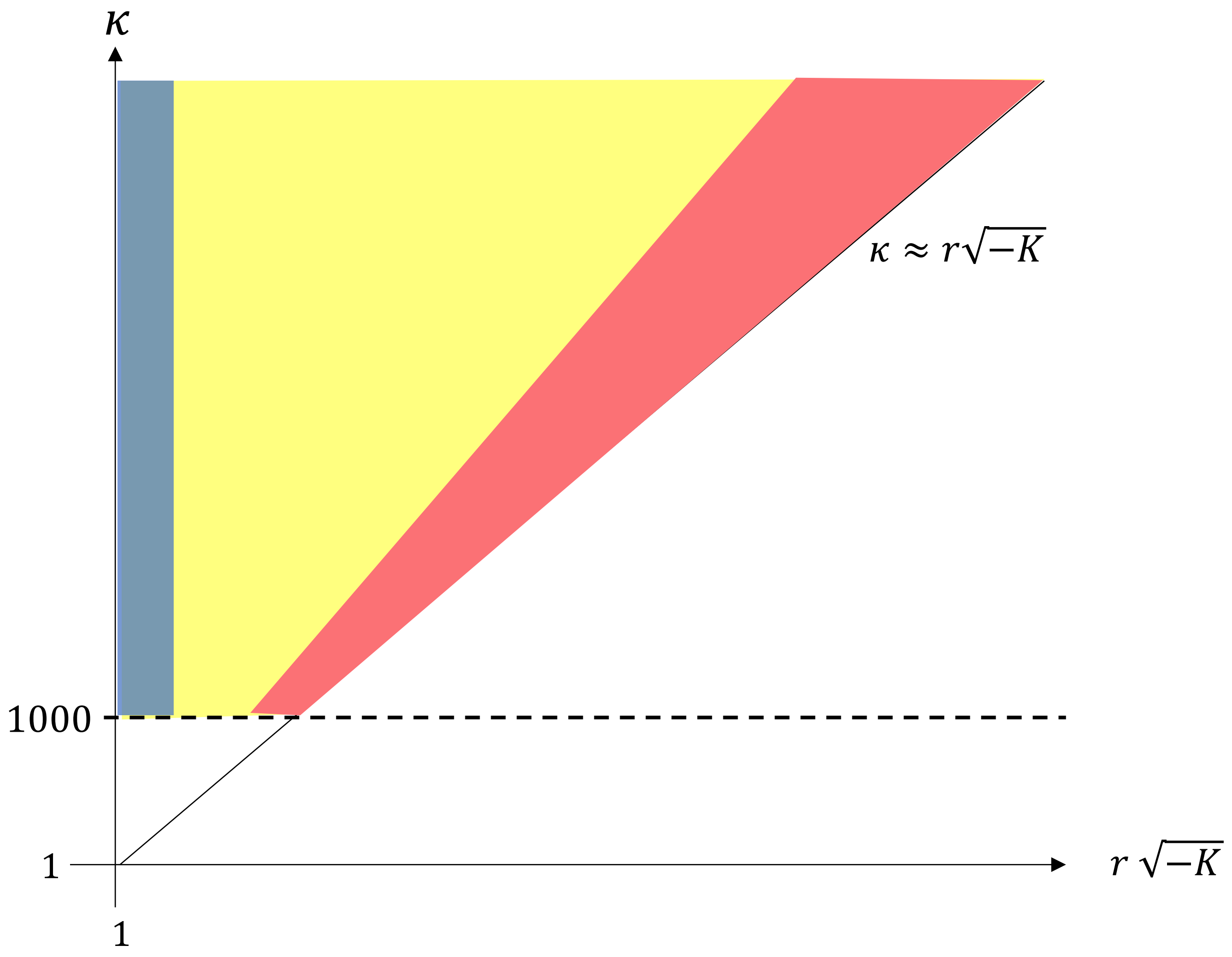
$$\mathrm{H}_{j,k}(x) = \sum_{m=1}^{k} h_{j,m}$$

One bump function $h_{j,m}$ is added for each query made by the algorithm.

Support of the bump $h_{j,m}$ is centered at the the query x_m.

Appendix

What we know (for hyperbolic spaces)



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Still, can prove the lower bound $\Omega\left(\frac{1}{n}\frac{\kappa}{\log \kappa}\right)$.

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Have the lower bound $\Omega\left(\frac{1}{\epsilon} \cdot \frac{1}{\log^3(\epsilon^{-1})}\right)$ for finding a point x with $f(x) - f(x^*) \leq \epsilon$.

Means a version of RGD is optimal.

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Compare with NAG, which uses at most $O\left(\frac{1}{\sqrt{\epsilon}}\right)$ queries in Euclidean spaces.

Applications

- Fréchet mean (intrinsic averaging on Hadamard spaces) (e.g., Karcher)
- Gaussian mixture models (Hosseini + Sra)
- Optimistic likelihoods for Gaussians (Nguyen et al.)
- Robust Covariance estimation (Weisel + Zhang, Franks + Moitra)
- Metric learning (Zadeh et al.)
- Variants on PCA (Tang + Allen) [MLEs for matrix normal models]
- Operator/tensor scaling (Allen Zhu et al., Burgisser et al.)
    - Brascamp-Lieb constants, computational complexity, polynomial identity testing, hardness of robust subspace recovery, etc.
- Tree-like embeddings (Bacak)
- Sampling on Riemannian manifolds (Goyal + Shetty)
- Landscape analysis (e.g., Ahn + Suarez)

Application: robust covariance estimation

IID samples $x_i \in \mathbb{R}^p, i = 1, \ldots, n$, coming from an elliptical distribution:

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where $\Sigma \succ 0$ is fixed (the shape matrix), u is a scalar r.v., and $v \sim \mathbb{S}^{p-1}$.

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Tyler's M-estimator for the shape matrix:

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Can also be derived as an MLE.

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Is g-convex for PD matrices (with affine-invariant metric).

→ new algorithms/analysis + analysis for Tyler's iterative procedure

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Is a specific instance of the operator scaling problem.

Sources: Weisel + Zhang, Franks + Moitra